BEAM DESIGN FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

DESIGN AID No. 6
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Introduction

Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the *Western Woods Use Book*, 4th edition, and are provided herein as a courtesy of Western Wood Products Association.

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Notations Relative to “Shear and Moment Diagrams”

- $E$ = modulus of elasticity, psi
- $I$ = moment of inertia, in.$^4$
- $L$ = span length of the bending member, ft.
- $l$ = span length of the bending member, in.
- $M$ = maximum bending moment, in.-lbs.
- $P$ = total concentrated load, lbs.
- $R$ = reaction load at bearing point, lbs.
- $V$ = shear force, lbs.
- $W$ = total uniform load, lbs.
- $w$ = load per unit length, lbs./in.
- $\Delta$ = deflection or deformation, in.
- $x$ = horizontal distance from reaction to point on beam, in.
Figure 1  Simple Beam – Uniformly Distributed Load

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = \frac{w \ell}{2} \]

\[ V_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = w \left( \frac{\ell}{2} - x \right) \]

\[ M_{\text{max}} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{w \ell^2}{8} \]

\[ M_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wx}{2} (\ell - x) \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{5w \ell^4}{384 EI} \]

\[ \Delta_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{wx}{24 EI} \left( \ell^3 - 2\ell x^2 + x^3 \right) \]

Figure 2  Simple Beam – Uniform Load Partially Distributed

\[ R_1 = V_1 \quad \text{(max when } a < c) \quad \ldots \ldots = \frac{wb \ell}{2 \ell} \left( 2c + b \right) \]

\[ R_2 = V_2 \quad \text{(max when } a > c) \quad \ldots \ldots = \frac{wb \ell}{2 \ell} \left( 2a + b \right) \]

\[ V_x \quad \text{(when } x > a \text{ and } (a + b)) \quad \ldots \ldots = R_1 - w (x - a) \]

\[ M_{\text{max}} \quad \text{at } x = a + \frac{R_1}{w} \quad \ldots \ldots = R_1 \left( a + \frac{R_1}{2w} \right) \]

\[ M_x \quad \text{(when } x < a) \quad \ldots \ldots \ldots \ldots \ldots = R_1 x \]

\[ M_x \quad \text{(when } x > a \text{ and } (a + b)) \quad \ldots \ldots = R_1 x - \frac{w}{2} (x - a)^2 \]

\[ M_x \quad \text{(when } x > (a + b)) \quad \ldots \ldots = R_2 (\ell - x) \]
Figure 3  Simple Beam – Uniform Load Partially Distributed at One End

\[ R_1 = V_1 = \frac{wa}{2\ell} (2\ell - a) \]

\[ R_2 = V_2 = \frac{wa}{2\ell} a \]

\[ V_x \text{ (when } x < a) = R_1 - wx \]

\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w} \right) = \frac{R_1}{2w} \]

\[ M_x \text{ (when } x < a) = R_1 x - \frac{wx^2}{2} \]

\[ M_x \text{ (when } x > a) = R_2 (\ell - x) \]

\[ \Delta_x \text{ (when } x < a) = \frac{wx}{24EI\ell} \left( a^2(2\ell - a)^2 - 2ax(2\ell - a) + ax^3 \right) \]

\[ \Delta_x \text{ (when } x > a) = \frac{wa^2(\ell - x)}{24EI\ell} \left( 4x\ell - 2x^2 - a^2 \right) \]

Figure 4  Simple Beam – Uniform Load Partially Distributed at Each End

\[ R_1 = V_1 = \frac{w_1 a (2\ell - a) + w_2 c^2}{2\ell} \]

\[ R_2 = V_2 = \frac{w_2 c (2\ell - c) + w_1 a^2}{2\ell} \]

\[ V_x \text{ (when } x < a) = R_1 - w_1 x \]

\[ V_x \text{ (when } x > a \text{ and } (a + b) \) = R_1 - w_1 a \]

\[ V_x \text{ (when } x > (a + b) \) = R_2 - w_2 (\ell - x) \]

\[ M_{\text{max}} \left( \text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a \right) = \frac{R_1^2}{2w_1} \]

\[ M_{\text{max}} \left( \text{at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c \right) = \frac{R_2^2}{2w_2} \]

\[ M_x \text{ (when } x < a) = R_1 x - \frac{w_1 x^2}{2} \]

\[ M_x \text{ (when } x > a \text{ and } (a + b) \) = R_1 x - \frac{w_1 a}{2} (2x - a) \]

\[ M_x \text{ (when } x > (a + b) \) = R_2 (\ell - x) - \frac{w_2 (\ell - x)^2}{2} \]
Figure 5  Simple Beam – Load Increasing Uniformly to One End

\[
R_1 = V_1 = \frac{W}{3} \\
R_2 = V_2 = \frac{2W}{3} \\
V_x = \frac{W}{3} - \frac{Wx^2}{\ell^2} \\
M_{max} \text{ at } x = \frac{\ell}{\sqrt{3}} = .57741 \ell = \frac{2W\ell}{9\sqrt{3}} = .1283W\ell \\
M_x = \frac{Wx}{3\ell^2} (\ell^2 - x^2) \\
\Delta_{max} \text{ at } x = \ell \left(1 - \frac{8}{15}\right) = .5193\ell \ell = .01304 \frac{W\ell^3}{EI} \\
\Delta_x = \frac{Wx}{180EI\ell^2} (3x^4 - 10\ell^2x^2 + 7\ell^4)
\]

Figure 6  Simple Beam – Load Increasing Uniformly to Center

\[
R = V = \frac{W}{2} \\
V_x \left( \text{ when } x < \frac{\ell}{2} \right) = \frac{W}{2\ell^2} (\ell^2 - 4x^2) \\
M_{max} \text{ (at center)} = \frac{W\ell}{6} \\
M_x \left( \text{ when } x < \frac{\ell}{2} \right) = \frac{Wx}{2} \left(\frac{1}{2} - \frac{2x^2}{3\ell^2}\right) \\
\Delta_{max} \text{ (at center)} = \frac{W\ell^3}{60EI} \\
\Delta_x = \frac{Wx}{480EI\ell^2} (5\ell^2 - 4x^2)^2
\]
Figure 7  Simple Beam – Concentrated Load at Center

\[ R = V \quad \ldots \ldots \ldots \ldots = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots = \frac{PE}{4} \]

\[ M_{i} \left( \text{when} \ x < \frac{\ell}{2} \right) \quad \ldots \ldots = \frac{Px}{2} \]

\[ \Delta_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots = \frac{PE^3}{48EI} \]

\[ \Delta_{i} \left( \text{when} \ x < \frac{\ell}{2} \right) \quad \ldots \ldots = \frac{Px}{48EI} (3\ell^2 - 4x^2) \]

Figure 8  Simple Beam – Concentrated Load at Any Point

\[ R_{1} = V_{1} \text{ (max when} \ a < b) \quad \ldots \ldots = \frac{Pb}{\ell} \]

\[ R_{2} = V_{2} \text{ (max when} \ a > b) \quad \ldots \ldots = \frac{Pa}{\ell} \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots = \frac{Pab}{\ell} \]

\[ M_{i} \left( \text{when} \ x < a \right) \quad \ldots \ldots = \frac{Pbx}{\ell} \]

\[ \Delta_{\text{max}} \left( \text{at} \ x = \left[ \frac{a(a + 2b)}{3}\right] \text{ when} \ a > b \right) \quad = \frac{Paba + 2b)\sqrt{3a(a + 2b)}}{27EIl} \]

\[ \Delta_{a} \text{ (at point of load)} \quad \ldots \ldots = \frac{Pa^2b^2}{3EIl} \]

\[ \Delta_{i} \left( \text{when} \ x < a \right) \quad \ldots \ldots = \frac{Px}{6EIl} (\ell^2 - b^2 - x^2) \]

\[ \Delta_{i} \left( \text{when} \ x > a \right) \quad \ldots \ldots = \frac{Pa(\ell - x)}{6EIl} (2\ell x - x^2 - a^2) \]
**Figure 9** Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

\[ R = V = P \]
\[ M_{\text{max}} \text{ (between loads)} = Pa \]
\[ M_x \text{ (when } x < a \text{)} = Px \]
\[ \Delta_{\text{max}} \text{ (at center)} = \frac{Pa}{24EI} (3\ell^2 - 4a^2) \]
\[ \Delta_x \text{ (when } x < a \text{)} = \frac{Px}{6EI} (3a - 3a^2 - x^2) \]
\[ \Delta_x \text{ (when } x > a \text{ and } (\ell - a) \) = \frac{Pa}{6EI} (3\ell^2 - 3x^2 - a^2) \]

**Figure 10** Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 \text{ (max when } a < b \text{)} = \frac{P}{\ell} (\ell - a + b) \]
\[ R_2 = V_2 \text{ (max when } a > b \text{)} = \frac{P}{\ell} (\ell - b + a) \]
\[ V_x \text{ (when } x > a \text{ and } (\ell - b) \text{) = } \frac{P}{\ell} (b - a) \]
\[ M_1 \text{ (max when } a > b \text{)} = R_1 a \]
\[ M_2 \text{ (max when } a < b \text{)} = R_2 b \]
\[ M_x \text{ (when } x < a \text{)} = R_1 x \]
\[ M_x \text{ (when } x > a \text{ and } (\ell - b) \text{) = } R_1 x - P(x - a) \]
Figure 11  Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 = \frac{P_1(\ell - a) + P_2b}{\ell} \]

\[ R_2 = V_2 = \frac{P_1a + P_2(\ell - b)}{\ell} \]

\[ V_x \text{ (when } x > a \text{ and } < (\ell - b) \text{)} = R_1 - P_1 \]

\[ M_1 (\text{max when } R_1 < P_1) = R_1a \]

\[ M_2 (\text{max when } R_2 < P_2) = R_2b \]

\[ M_s (\text{when } x < a) = R_1x \]

\[ M_s (\text{when } x > a \text{ and } < (\ell - b)) = R_1x - P_1(x - a) \]

Figure 12  Cantilever Beam – Uniformly Distributed Load

\[ R = V = w\ell \]

\[ V_x = wx \]

\[ M_{\text{max}} (\text{at fixed end}) = \frac{w\ell^2}{2} \]

\[ M_s = \frac{wx^2}{2} \]

\[ \Delta_{\text{max}} (\text{at free end}) = \frac{w\ell^4}{8EI} \]

\[ \Delta_x = \frac{w}{24EI} (x^4 - 4\ell^3x + 3\ell^4) \]
Figure 13  Cantilever Beam – Concentrated Load at Free End

\[ R = V \quad \ldots \ldots \ldots \ldots \quad = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots \ldots \quad = P\ell \]
\[ M_x \quad \ldots \ldots \ldots \ldots \quad = Px \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots \ldots \quad = \frac{P\ell^4}{3EI} \]
\[ \Delta_x \quad \ldots \ldots \ldots \ldots \quad = \frac{P}{6EI}(2\ell^3 - 3\ell^2x + x^3) \]

Figure 14  Cantilever Beam – Concentrated Load at Any Point

\[ R = V \quad \ldots \ldots \ldots \ldots \quad = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots \ldots \quad = Pb \]
\[ M_x \text{ (when } x > a \text{)} \quad \ldots \ldots \ldots \ldots \quad = P(x - a) \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots \ldots \quad = \frac{Pb^2}{6EI}(3\ell - b) \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots \ldots \quad = \frac{Pb^3}{3EI} \]
\[ \Delta_x \text{ (when } x < a \text{)} \quad \ldots \ldots \ldots \ldots \quad = \frac{Pb^2}{6EI}(3\ell - 3x - b) \]
\[ \Delta_x \text{ (when } x > a \text{)} \quad \ldots \ldots \ldots \ldots \quad = \frac{P(\ell - x)^2}{6EI}(3b - \ell + x) \]
Figure 15  Beam Fixed at One End, Supported at Other – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{3w\ell}{8} \]
\[ R_2 = V_2 = \frac{5w\ell}{8} \]
\[ V_x = R_1 - wx \]
\[ M_{\text{max}} = \frac{w\ell^2}{8} \]
\[ M_1 \left( \text{at } x = \frac{3\ell}{8} \right) = \frac{9w\ell^2}{128} \]
\[ M_x = R_1x - \frac{wx^2}{2} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{16} (1 + \sqrt{33}) = .4215\ell \right) = \frac{w\ell^4}{185EI} \]
\[ \Delta_x = \frac{wx}{48EI} (\ell^3 - 3\ell x^2 + 2x^3) \]

Figure 16  Beam Fixed at One End, Supported at Other – Concentrated Load at Center

\[ R_1 = V_1 = \frac{5P}{16} \]
\[ R_2 = V_2 = \frac{11P}{16} \]
\[ M_{\text{max}} \text{ (at fixed end)} = \frac{3P\ell}{16} \]
\[ M_1 \text{ (at point of load)} = \frac{5P\ell}{32} \]
\[ M_x \text{ (when } x < \frac{\ell}{2} \) = \frac{5Px}{16} \]
\[ M_x \text{ (when } x > \frac{\ell}{2} \) = \frac{P}{16} \left( \frac{\ell}{2} - \frac{11x}{16} \right) \]
\[ \Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{16} \left[ 1 + \frac{1}{\sqrt{5}} \right] = .4472\ell \right) = \frac{P\ell^3}{48EI\sqrt{5}} = .009317 \frac{P\ell^3}{EI} \]
\[ \Delta_x \text{ (at point of load)} = \frac{7P\ell^3}{768EI} \]
\[ \Delta_x \text{ (when } x < \frac{\ell}{2} \) = \frac{Px}{96EI} (3\ell^2 - 5x^2) \]
\[ \Delta_x \text{ (when } x > \frac{\ell}{2} \) = \frac{P}{96EI} (x - \ell)^2(11x - 2\ell) \]
Figure 17  Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point

\[ R_1 = V_1 = \frac{PB}{2L} (a + 2L) \]
\[ R_2 = V_2 = \frac{Pa}{2L} (3L^2 - a^2) \]
\[ M_1 \text{ (at point of load)} = R_1 a \]
\[ M_2 \text{ (at fixed end)} = \frac{Pab}{2L^3} (a + \ell) \]
\[ M_3 \text{ (when } x < a) = R_1 x \]
\[ M_4 \text{ (when } x > a) = R_1 x - P(x - a) \]
\[ \Delta_{max} \text{ (when } a < 0.414L \text{ at } x = \ell - \frac{\ell^2 + a^2}{3L^2 - a^2} = \frac{Pa}{3EI} \left(\frac{\ell^2 - a^2}{3L^2 - a^2}\right)^2 \]
\[ \Delta_{max} \text{ (when } a > 0.414L \text{ at } x = \ell - \frac{a}{2\ell + a} = \frac{Pab^2}{6EI} \left(\frac{a}{2\ell + a}\right)^3 \]
\[ \Delta_1 \text{ (at point of load)} = \frac{Pa^2b}{12EI} (3L^2 + a) \]
\[ \Delta_2 \text{ (when } x < a) = \frac{Pb^2}{12EI} (3aL^2 - 2x^2 - ax^2) \]
\[ \Delta_3 \text{ (when } x > a) = \frac{Pa}{12EI} (\ell - x)^2 (3L^2 x - a^2 x - 2a^2 \ell) \]

Figure 18  Beam Overhanging One Support – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{w}{2\ell} (\ell^2 - a^2) \]
\[ R_2 = V_2 + V_1 = \frac{w}{2\ell} (\ell + a)^2 \]
\[ V_2 = wa \]
\[ V_3 = \frac{w}{2\ell} (\ell^2 + a^2) \]
\[ V_4 \text{ (between supports)} = R_1 - wx \]
\[ V_{s1} \text{ (for overhang)} = wa(x - \ell) \]
\[ M_1 \text{ (at } x = \frac{\ell}{2} \left[1 - \frac{a^2}{\ell^2}\right]\} = \frac{w}{8\ell^2} (\ell + a)^2 (\ell - a)^2 \]
\[ M_2 \text{ (at } R_2) = \frac{wa^2}{2} \]
\[ M_3 \text{ (between supports)} = \frac{wx}{2\ell} (\ell^2 - a^2 - x\ell) \]
\[ M_{s1} \text{ (for overhang)} = \frac{w}{2} (a - x_s)^2 \]
\[ \Delta_4 \text{ (between supports)} = \frac{wx}{24EI} (\ell^4 - 2\ell^2 x^2 + \ell x^3 - 2a^2 \ell^2 + 2a^2 x^2) \]
\[ \Delta_{s1} \text{ (for overhang)} = \frac{wx_s}{24EI} (4a^2 \ell - \ell^3 + 6a^2 x_s - 4ax_s^2 + x_s^3) \]
Figure 19  Beam Overhanging One Support – Uniformly Distributed Load on Overhang

\[ R_1 = V_1 = \frac{wa^2}{2\ell} \]
\[ R_2 = V_1 + V_2 = \frac{wa}{2\ell} (2\ell + a) \]
\[ V_2 = \frac{wa}{2\ell} \]
\[ V_{x_1} \text{ (for overhang)} = w(a - x_1) \]
\[ M_{\text{max}} \text{ (at } R_1) = \frac{wa^2}{2} \]
\[ M_x \text{ (between supports)} = \frac{wa^2 x}{2\ell} \]
\[ M_{x_1} \text{ (for overhang)} = \frac{w}{2} (a - x_1)^2 \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{wa^2 \ell^3}{18\sqrt{3}EI} = 0.03208 \frac{wa^2 \ell^3}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{wa^3 (4\ell + 3a)}{24EI} \]
\[ \Delta_x \text{ (between supports)} = \frac{wa^2 x}{12EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{wa_x (4a^2 \ell + 6a^2 x_1 - 4a^2 x_1^2 + x_1^4)}{24EI} \]

Figure 20  Beam Overhanging One Support – Concentrated Load at End of Overhang

\[ R_1 = V_1 = \frac{Pa}{\ell} \]
\[ R_2 = V_1 + V_2 = \frac{P}{\ell} (\ell + a) \]
\[ V_2 = \frac{P}{\ell} \]
\[ M_{\text{max}} \text{ (at } R_1) = Pa \]
\[ M_x \text{ (between supports)} = \frac{Pax}{\ell} \]
\[ M_{x_1} \text{ (for overhang)} = P(a - x_1) \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{Pa^2 \ell}{9\sqrt{3}EI} = 0.06415 \frac{Pa^2 \ell}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{Pa^2}{3EI} (\ell + a) \]
\[ \Delta_x \text{ (between supports)} = \frac{Pax}{6EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} = \frac{Pax_1}{6EI} (2a\ell + 3ax_1 - x_1^2) \]
Figure 21  Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

\[ R_1 = V_1 \text{ (max when } a < b) \]
\[ R_2 = V_2 \text{ (max when } a > b) \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{Pab}{\ell} \]
\[ M_x \text{ (when } x < a) = \frac{Pbx}{\ell} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \frac{(a + 2b)}{3} \text{ when } a > b \right) = \frac{Pab(a + 2b)\sqrt{3(a + 2b)}}{27EI\ell} \]
\[ \Delta_a \text{ (at point of load)} = \frac{Pa^2b^2}{3EI\ell} \]
\[ \Delta_x \text{ (when } x < a) = \frac{Pbx^2}{6EI\ell} \left( \ell^2 - b^2 - x^2 \right) \]
\[ \Delta_x \text{ (when } x > a) = \frac{Pa(\ell - x)}{6EI\ell} \left( 2\ell x - x^2 - a^2 \right) \]
\[ \Delta_x = \frac{Pabx}{6EI\ell} (\ell + a) \]

Figure 22  Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

\[ R_1 = \frac{w\ell (\ell - 2c)}{2b} \]
\[ R_2 = \frac{w\ell (\ell - 2a)}{2b} \]
\[ V_1 = wa \]
\[ V_2 = R_1 - V_1 \]
\[ V_4 = R_2 - V_4 \]
\[ V_3 = V_1 - wx_1 \]
\[ V_c \text{ (when } x = \ell) = R_1 - w(a + x_1) \]
\[ V_m \text{ (when } a < c) = R_2 - wc \]
\[ M_1 = -\frac{wa^2}{2} \]
\[ M_2 = -\frac{wc^2}{2} \]
\[ M_3 = R_1 \left( \frac{R_1 - a}{2w} \right) \]
\[ M_x \text{ (max when } x = \frac{R_1}{w} - a) = R_1 x - \frac{w(a + x)^2}{2} \]
**Figure 23** Beam Fixed at Both Ends – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]
\[ V_x = w\left(\frac{\ell}{2} - x\right) \]
\[ M_{\text{max}} \text{ (at ends)} = \frac{w\ell^2}{12} \]
\[ M_0 \text{ (at center)} = \frac{w\ell^2}{24} \]
\[ M_x = \frac{w}{12}(6\ell x - \ell^2 - 6x^2) \]
\[ \Delta_{\text{max}} \text{ (at center)} = \frac{w\ell^4}{384EI} \]
\[ \Delta_x = \frac{wx^2}{24EI}(\ell - x)^2 \]

**Figure 24** Beam Fixed at Both Ends – Concentrated Load at Center

\[ R = V = \frac{P}{2} \]
\[ M_{\text{max}} \text{ (at center and ends)} = \frac{P\ell}{8} \]
\[ M_x \left( \text{ when } x < \frac{\ell}{2} \right) = \frac{P}{8}(4x - \ell) \]
\[ \Delta_{\text{max}} \text{ (at center)} = \frac{P\ell^3}{192EI} \]
\[ \Delta_x \left( \text{ when } x < \frac{\ell}{2} \right) = \frac{Px^2}{48EI}(3\ell - 4x) \]
**Figure 25**  Beam Fixed at Both Ends – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) = \frac{Pb^2}{\ell} (3a + b) \]
\[ R_2 = V_2 \text{ (max when } a > b) = \frac{Pa^2}{\ell^2} (a + 3b) \]
\[ M_1 \text{ (max when } a < b) = \frac{Pab^2}{\ell^2} \]
\[ M_2 \text{ (max when } a > b) = \frac{Pa^2b}{\ell^2} \]
\[ M_a \text{ (at point of load)} = \frac{2Pa^2b^2}{\ell^4} \]
\[ M_x \text{ (when } x < a) = R_1x - \frac{Pab^2}{\ell^2} \]
\[ \Delta_{max} \left( \text{when } a > b \text{ at } x = \frac{2a\ell}{3a + b} \right) = \frac{2Pa^2b^2}{3EI(3a + b)^2} \]
\[ \Delta_a \text{ (at point of load)} = \frac{Pab^3}{3EI\ell} \]
\[ \Delta_x \text{ (when } x < a) = \frac{Pb^2x^2}{6EIo} (3a\ell - 3ax - bx) \]

**Figure 26**  Continuous Beam – Two Equal Spans – Uniform Load on One Span

\[ R_1 = V_1 = \frac{7}{16} w\ell \]
\[ R_2 = V_2 + V_1 = \frac{5}{8} w\ell \]
\[ R_3 = V_1 = -\frac{1}{16} w\ell \]
\[ V_2 = \frac{9}{16} w\ell \]
\[ M_{max} \left( \text{at } x = \frac{7}{16} \ell \right) = \frac{49}{512} w\ell^2 \]
\[ M_i \text{ (at support } R_2) = \frac{1}{16} w\ell^2 \]
\[ M_x \text{ (when } x < \ell) = \frac{w\ell x}{16} (7\ell - 8x) \]
Figure 27  Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

\[ R_1 = V_1 = \frac{13}{32} P \]
\[ R_2 = V_2 + V_1 = \frac{11}{16} P \]
\[ R_3 = V_3 = -\frac{3}{32} P \]
\[ V_2 = \frac{19}{32} P \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{13}{64} P\ell \]
\[ M_1 \text{ (at support } R_2) = \frac{3}{32} P\ell \]

Figure 28  Continuous Beam – Two Equal Spans – Concentrated Load at Any Point

\[ R_1 = V_1 = \frac{Pb}{4\ell^3} \left(4\ell^2 - a(\ell + a)\right) \]
\[ R_2 = V_2 + V_1 = \frac{Pa}{2\ell^3} \left(2\ell^2 + b(\ell + a)\right) \]
\[ R_3 = V_3 = -\frac{Pab}{4\ell^3} (\ell + a) \]
\[ V_2 = \frac{Pa}{4\ell^3} \left(4\ell^2 + b(\ell + a)\right) \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{Pab}{4\ell^3} \left(4\ell^2 - a(\ell + a)\right) \]
\[ M_1 \text{ (at support } R_2) = \frac{Pab}{4\ell^3} (\ell + a) \]
Figure 29  Continuous Beam – Two Equal Spans – Uniformly Distributed Load

\[ R_1 = V_1 = R_3 = V_3 = \frac{3wl}{8} \]
\[ R_2 = \frac{10wl}{8} \]
\[ V_2 = V_{\text{max}} = \frac{5wl}{8} \]
\[ M_1 = \frac{wl^2}{8} \]
\[ M_2 \left( \text{at} \frac{3l}{8} \right) = \frac{9wl^2}{128} \]
\[ \Delta_{\text{max}} \text{ (at 0.4215} \ell, \text{ approx. from } R_1 \text{ and } R_3) \ldots = \frac{wl^4}{185EI} \]

Figure 30  Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

\[ R_1 = V_1 = R_3 = V_3 = \frac{5P}{16} \]
\[ R_2 = 2V_2 = \frac{11P}{8} \]
\[ V_2 = P - R_1 = \frac{11P}{16} \]
\[ V_{\text{max}} = V_2 \]
\[ M_1 = -\frac{3Pl}{16} \]
\[ M_2 = \frac{5Pl}{32} \]
\[ M_x \text{ (when } x < a) = R_1x \]
Figure 31  Continuous Beam – Two Unequal Spans – Uniformly Distributed Load

\[ R_1 = \frac{M_1}{\ell_1} + \frac{w\ell_1}{2} \]
\[ R_2 = w\ell_1 + w\ell_2 - R_1 - R_3 \]
\[ R_1 = V_4 = \frac{M_1}{\ell_2} + \frac{w\ell_1}{2} \]
\[ V_1 = R_1 - R_1 \]
\[ V_2 = w\ell_1 - R_1 \]
\[ V_4 = R_4 \]
\[ M_4 = \frac{w\ell_1^3 + w\ell_2^3}{8(\ell_1 + \ell_2)} \]
\[ M_{x1} \left( \text{when } x_1 = \frac{R_1}{w} \right) = R_1 x_1 - \frac{w\ell_1 x_1^2}{2} \]
\[ M_{x2} \left( \text{when } x_2 = \frac{R_3}{w} \right) = R_3 x_2 - \frac{w\ell_2 x_2^2}{2} \]

Figure 32  Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed

\[ R_1 = \frac{M_1}{\ell_1} + \frac{P_1}{2} \]
\[ R_2 = P_1 + P_2 - R_1 - R_3 \]
\[ R_3 = \frac{M_1}{\ell_2} + \frac{P_3}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = P_1 - R_1 \]
\[ V_4 = P_3 - R_3 \]
\[ V_4 = R_4 \]
\[ M_4 = -\frac{3}{16} \left( \frac{P_1 \ell_1^2 + P_3 \ell_2^2}{\ell_1 + \ell_2} \right) \]
\[ M_{x1} = R_1 a \]
\[ M_{x2} = R_3 b \]